Some Sums are not Rational Functions of R, r, and s

Stanley Rabinowitz
12 Vine Brook Road
Westford, MA 01886-4212 USA

Let R, r, and s denote the circumradius, inradius, and semiperimeter of a triangle with angles A, B, and C. In problem 652 of this journal [2], W. J. Blundon pointed out the well-known formulae

$$\sum \sin A = \frac{s}{R}$$
$$\sum \cos A = \frac{R+r}{R}$$
$$\sum \tan A = \frac{2rs}{s^2-4R^2-4Rr-r^2}$$
$$\sum \tan \frac{A}{2} = \frac{4R+r}{s}$$

where the sums are cyclic over the angles of the triangle. He asked if there were similar formulae for $\sum \sin A/2$ and $\sum \cos A/2$. All the solutions received were very complicated. Murray Klamkin [3] pointed out that Anders Bager in [1] tacitly implied there were no known simple $R-r-s$ representations for the following triangle functions:

$$\begin{gathered}\sum \sin \tfrac{A}{2}, \quad \sum \sin \tfrac{B}{2} \sin \tfrac{C}{2}, \quad \sum \csc \tfrac{A}{2}, \quad \sum \csc \tfrac{B}{2} \csc \tfrac{C}{2}, \\ \sum \cos \tfrac{A}{2}, \quad \sum \cos \tfrac{B}{2} \cos \tfrac{C}{2}, \quad \sum \sec \tfrac{A}{2}, \quad \sum \sec \tfrac{B}{2} \sec \tfrac{C}{2}.\end{gathered} \tag{1}$$

Klamkin went on to conjecture that these sums cannot be expressed as rational functions of R, r, and s. (A *rational function* is the quotient of two polynomials.) This conjecture is made plausible by the fact that compendiums of such formulae (such as chapter 4 of [4]) do not include values for these particular sums. In this note, we will prove Klamkin's conjecture.

Theorem. $\sum \sin A/2$ and $\sum \cos A/2$ can not be expressed as rational functions of R, r, and s.

Proof. Consider a triangle ABC with sides $BC = 13$, $CA = 14$, and $AB = 15$. This triangle has area 84, semiperimeter 21, inradius 4, and circumradius 65/8. From the Law of Cosines, we can easily compute the cosines of the angles, finding

$$\begin{aligned} \cos A &= \tfrac{3}{5} & \qquad & & \sin A &= \tfrac{4}{5} \\ \cos B &= \tfrac{33}{65} & & \text{and} & \sin B &= \tfrac{56}{65} \\ \cos C &= \tfrac{5}{13} & & & \sin C &= \tfrac{12}{13}. \end{aligned}$$

Reprinted from Crux Mathematicorum, 16(1990)1–3

From the half-angle formulae, we find that

$$\begin{array}{lll} \sin\frac{A}{2}=\frac{1}{\sqrt{5}} & \sin\frac{B}{2}=\frac{4}{\sqrt{65}} & \sin\frac{C}{2}=\frac{2}{\sqrt{13}} \\ \cos\frac{A}{2}=\frac{2}{\sqrt{5}} & \cos\frac{B}{2}=\frac{7}{\sqrt{65}} & \cos\frac{C}{2}=\frac{3}{\sqrt{13}} \\ \sec\frac{A}{2}=\frac{1}{2}\sqrt{5} & \sec\frac{B}{2}=\frac{1}{7}\sqrt{65} & \sec\frac{C}{2}=\frac{1}{3}\sqrt{13} \\ \csc\frac{A}{2}=\sqrt{5} & \csc\frac{B}{2}=\frac{1}{4}\sqrt{65} & \csc\frac{C}{2}=\frac{1}{2}\sqrt{13}. \end{array}$$

We thus see that

$$\sum \sin\frac{A}{2}=\frac{1}{\sqrt{5}}+\frac{4}{\sqrt{65}}+\frac{2}{\sqrt{13}}$$

which is irrational. This shows that $\sum \sin A/2$ cannot be a rational function of R, r, and s, for if it were, then in this particular case, its numeric value would be rational (since R, r, and s are rational in this case), a contradiction. Similarly, $\sum \cos A/2$ cannot be a rational function of R, r, and s, because that would be contradicted by this particular case, in which

$$\sum \cos\frac{A}{2}=\frac{2}{\sqrt{5}}+\frac{7}{\sqrt{65}}+\frac{3}{\sqrt{13}}$$

is also irrational. □

A similar calculation and argument shows that none of the expressions in display (1) can be expressed as rational functions of R, r, and s. In fact, the same argument shows further that none of these expressions can be expressed as rational functions of R, r, s, a, b, c, and K, where a, b, and c are the lengths of the sides of the triangle and K is its area.

In many cases, similar results can be shown using simpler examples. For example, let m_a, m_b, m_c denote the lengths of the medians of a triangle. Using a 3-4-5 right triangle, I showed in [5] that there is no rational function, M, of a, b, and c such that each of m_a, m_b, m_c can be expressed as rational functions of a, b, c, and $\sqrt{M}$.

As an exercise, the reader can prove that $\sin x/2$ and $\cos x/2$ can not be expressed as rational functions of $\sin x$ and $\cos x$. It is well known that $\tan x/2$ can be so expressed, namely

$$\tan\frac{x}{2}=\frac{\sin x}{1+\cos x}.$$

REFERENCES

[1] Anders Bager, "A Family of Goniometric Inequalities", *Publikacije Elektrotehničkog Fakulteta Univerziteta U Beogradu, Serija: Matematika i Fizika.* **Nos. 338–352** (1971)5–25.
[2] W. J. Blundon, "Problem 652", *Crux Mathematicorum.* **7**(1981)179.
[3] M. S. Klamkin, "Comment on Problem 652", *Crux Mathematicorum.* **8**(1982)189.

[4] D. S. Mitrinović, J. E. Pečarić, and V. Volenec, *Recent Advances in Geometric Inequalities*. Kluwer Academic Publishers. Boston: 1989.

[5] Stanley Rabinowitz, "On the Computer Solution of Symmetric Homogeneous Triangle Inequalities" in Proceedings of the ACM-SIGSAM 1989 International Symposium on Symbolic and Algebraic Computation (ISSAC '89), 272–286.