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A Triad of Circles Associated with a Triangle

Ercole Suppa[a] and Stanley Rabinowitz[b]
[a] Via B. Croce 54, 64100 Teramo, Italia
e-mail: ercolesuppa@gmail.com
web: http://www.esuppa.it/
[b] 545 Elm St Unit 1, Milford, New Hampshire 03055, USA
e-mail: stan.rabinowitz@comcast.net[2]
web: http://www.StanleyRabinowitz.com/

Abstract. We study some properties of a triad of circles associated with a triangle. Each circle is inside the triangle, tangent to two sides of the triangle, and externally tangent to the circle on the third side as diameter. In particular, we find a nice relation involving the radii of the inner and outer Apollonius circles of the three circles in the triad.

Keywords. Paasche point, Apollonius circle, barycentric coordinates, Mathematica.

Mathematics Subject Classification (2020). 51-02, 51M04.

1. Introduction

Notation. Throughout this paper, we will use the following notation, where $\triangle ABC$ is a fixed acute triangle in the plane. We let $a = BC$, $b = CA$, $c = AB$, r is the inradius of $\triangle ABC$, R is the circumradius of $\triangle ABC$, $p = \frac{a+b+c}{2}$, $\Delta = [ABC]$ is the area of the triangle, and $S = 2\Delta$. We also let I denote the incenter of $\triangle ABC$.

The semicircle erected inwardly on side BC will be named ω_a as shown in Figure 1 (left). Semicircles ω_b and ω_c are defined similarly. The circle inside $\triangle ABC$, tangent to sides AB and AC, and externally tangent to semicircle ω_a will be named γ_a. Circles γ_b and γ_c are defined similarly. The radii of circles γ_a, γ_b, and γ_c are denoted by ρ_a, ρ_b, and ρ_c, respectively. The centers of these circles are named D, E, and F, respectively, as shown in Figure 1 (right).

For purposes of this paper, these three circles will be called the *triad of circles* associated with $\triangle ABC$.

[2]Corresponding author

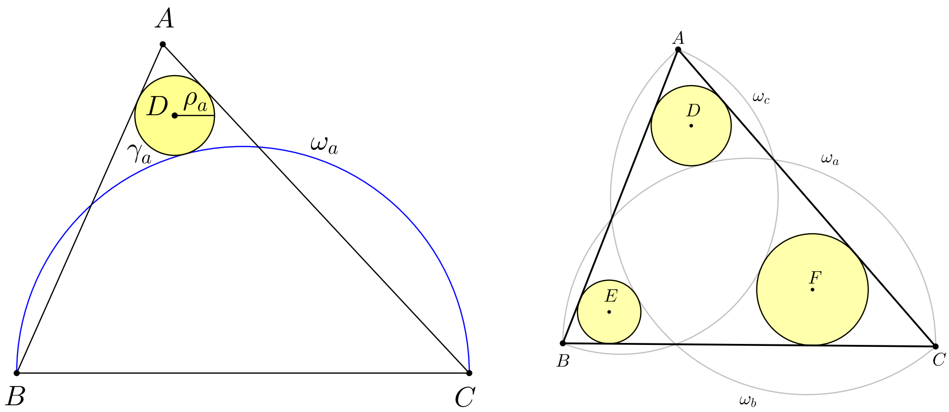

FIGURE 1.

This triad of circles appears in a Sangaku described in [3] and reprinted in [6, problem 6]. The statement in the Sangaku is given as Theorem 1.1.

Theorem 1.1. *For the triad of circles associated with $\triangle ABC$, we have*

$$r = \frac{1}{2}\left(\rho_a + \rho_b + \rho_c + \sqrt{\rho_a^2 + \rho_b^2 + \rho_c^2}\right).$$

A proof of this result can be found in [10]. A variant on this result when the triangle is not acute can also be found in [10].

It is the purpose of this paper to give other properties of such a triad of circles.

2. Known Results

Before giving new results, we summarize some of the properties already known about the triad of circles. The following five theorems come from [10].

Theorem 2.1. *For the triad of circles associated with $\triangle ABC$, we have*

$$\rho_a = r\left(1 - \tan\frac{A}{2}\right).$$

Similar formulas hold for ρ_b and ρ_c.

Theorem 2.2. *Let P and Q be the feet of the perpendiculars from D and I to side AC, respectively. Then $PQ = IQ$. (Figure 2)*

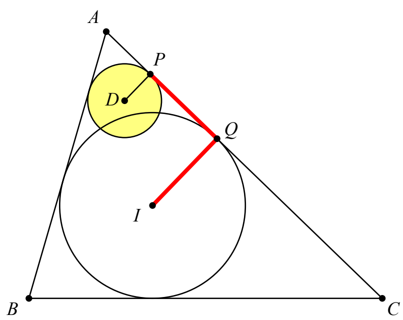

FIGURE 2. red lengths are equal

Theorem 2.3. *The lengths of the common external tangents between any two circles of the triad are equal. The common length is $2r$. (Figure 3)*

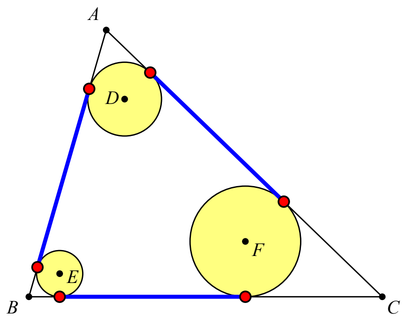

FIGURE 3. blue lengths are equal

Theorem 2.4. *The six points of contact of the triad of circles associated with $\triangle ABC$ lie on a circle with center I and radius $r\sqrt{2}$. (Figure 4)*

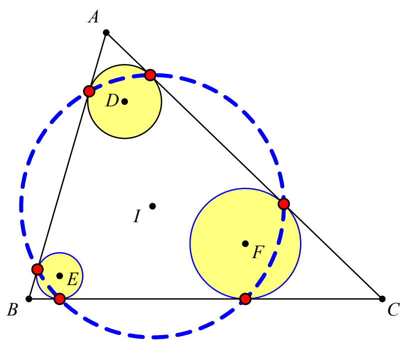

FIGURE 4.

This circle will be called the *contact circle*.

The following corollary follows immediately from Theorem 2.4.

Corollary 2.1. *In Figure 5 showing the contact circle and the incircle, the green area is equal to the blue area.*

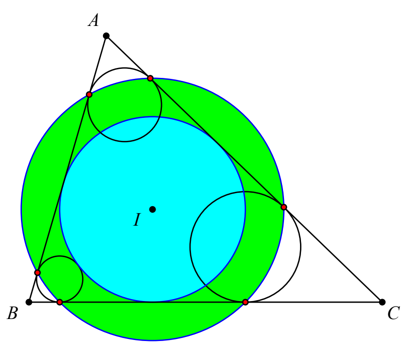

FIGURE 5. green area = blue area

Theorem 2.5. *For the triad of circles associated with $\triangle ABC$, we have*

$$\rho_a^2 + \rho_b^2 + \rho_c^2 = \frac{r^2(p - 4R - r)^2}{p^2}. \tag{1}$$

The following result comes from [4] where it is stated that the result is due to Tomasz Cieśla.

Theorem 2.6. *Let T_a, T_b, and T_c be the touch points of the circles in the triad with their corresponding semicircles as shown in Figure 6. Then AT_a, BT_b, and CT_c are concurrent.*

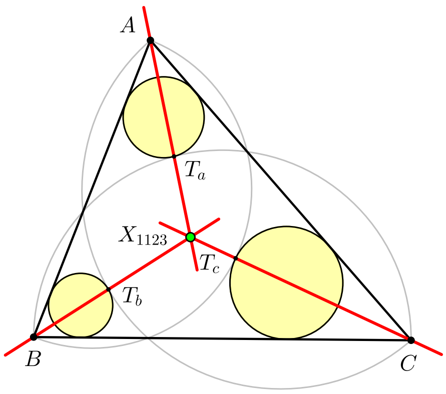

Figure 6.

The point of concurrence is catalogued as point X_{1123} in the Encyclopedia of Triangle Centers [4]. Since reference [4] does not include a proof of this result, we will give our own proof later in Section 5 of this paper.

The point X_{1123} is known as the *Paasche point* of the triangle because Paasche proved the following result in [7].

Theorem 2.7. *Congruent circles with centers A_1 and A_2 touch each other externally at point A' outside $\triangle ABC$. Circle (A_1) is tangent to AB and BC. Circle (A_2) is tangent to AC and BC. Points B' and C' are defined similarly (Figure 7). Then AA', BB', and CC' are concurrent.*

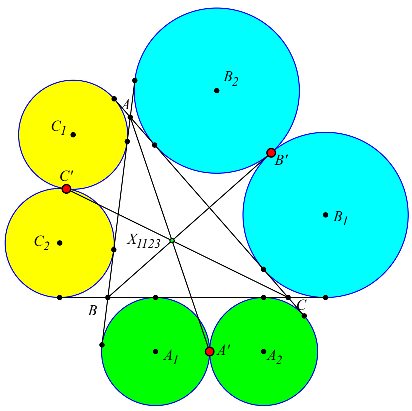

Figure 7.

Remark. The same result is true if the pairs of congruent circles are inside the triangle instead of outside. Figure 8 illustrates this. (Only the two congruent circles tangent to side BC are shown.) This result comes from [11, Art. 3.5.4, ex. 4c].

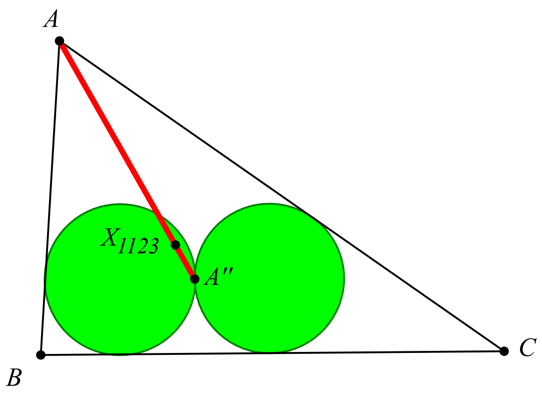

Figure 8.

The Paasche point can also be characterized as follows according to [2].

Theorem 2.8. *In $\triangle ABC$, let ω_a, ω_b, and ω_c be the circles constructed using sides BC, CA, and AB, respectively, as diameters. Let Ω be the circle internally tangent to ω_a, ω_b, and ω_c. Let A' be the touch point between ω_a and Ω. Points B' and C' are defined similarly (Figure 9). Then AA', BB', and CC' are concurrent at X_{1123}, the Paasche point of $\triangle ABC$.*

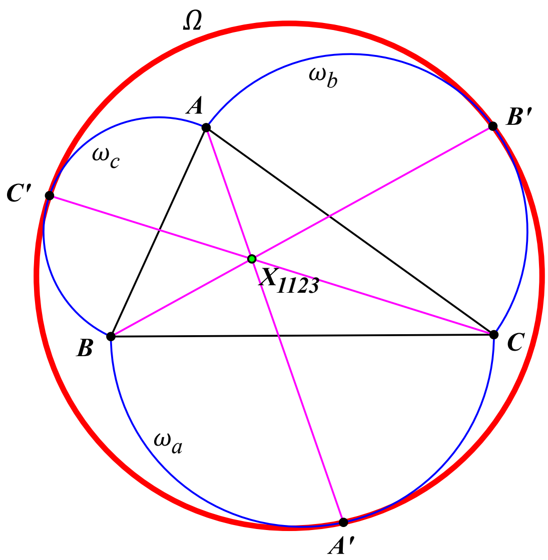

Figure 9.

A circle that is tangent to three given circles is called an *Apollonius circle* of those three circles.

If all three circles lie inside an Apollonius circle, then the Apollonius circle is called the *outer Apollonius circle* of the three circles. The outer Apollonius circle surrounds the three circles and is internally tangent to all three.

If all three circles lie outside an Apollonius circle, then the Apollonius circle is called the *inner Apollonius circle* of the three circles. The inner Apollonius circle will either be internally tangent to the three given circles or it will be externally tangent to all the circles. The following theorem comes from [5].

Theorem 2.9. *For the triad of circles associated with $\triangle ABC$, the inner Apollonius circle of γ_a, γ_b, γ_c, is internally tangent to the inner Apollonius circle of ω_a, ω_b, ω_c (Figure 10).*

Remark. The inner Apollonius circle of γ_a, γ_b, γ_c is known as the 1st Miyamoto-Moses-Apollonius circle and the outer Apollonius circle of γ_a, γ_b, γ_c is known as the 2nd Miyamoto-Moses-Apollonius circle (see [5]).

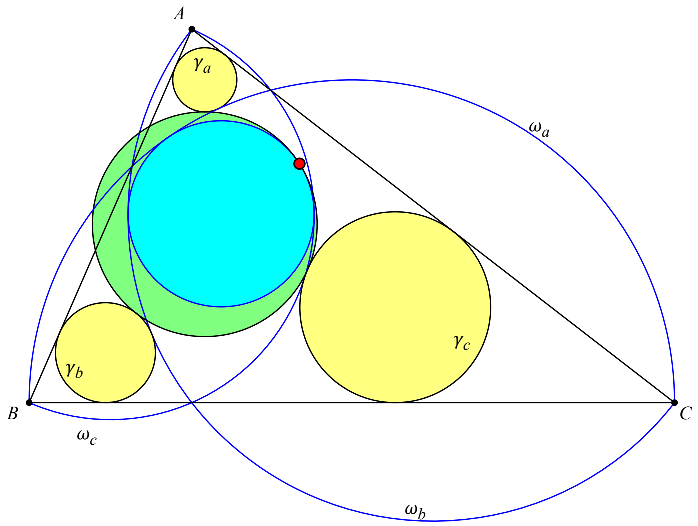

FIGURE 10.

3. METRIC RELATIONSHIPS INVOLVING ρ_a, ρ_b, ρ_c

In addition to Theorem 1.1 and Theorem 2.5, the following symmetric relationship involving ρ_a, ρ_b, and ρ_c holds.

Theorem 3.1. *For the triad of circles associated with $\triangle ABC$, we have*

$$\rho_a\rho_b + \rho_b\rho_c + \rho_c\rho_a - 2r(\rho_a + \rho_b + \rho_c) + 2r^2 = 0.$$

Proof. From Theorem 2.1, we have $\rho_a = r(1 - \tan\frac{A}{2})$, $\rho_b = r(1 - \tan\frac{B}{2})$, and $\rho_c = r(1 - \tan\frac{C}{2})$. Substituting these values into the expression

$$\rho_a\rho_b + \rho_b\rho_c + \rho_c\rho_a - 2r(\rho_a + \rho_b + \rho_c) + 2r^2$$

and simplifying (using MATHEMATICA), shows that the expression is equal to

$$-r^2\cos\left(\frac{A+B+C}{2}\right)\sec\frac{A}{2}\sec\frac{B}{2}\sec\frac{C}{2}.$$

Since $A + B + C = \pi$, this expression is equal to 0. □

Theorem 3.2. *For the triad of circles associated with $\triangle ABC$, we have*

$$p(r - \rho_a)(r - \rho_b)(r - \rho_c) = r^4.$$

Proof. This result follows from Theorem 2.1 and the trigonometric identity

$$\tan\frac{A}{2}\tan\frac{B}{2}\tan\frac{C}{2}=\frac{r}{p}$$

which comes from [1, p. 358]. □

Theorem 3.3. *For the triad of circles associated with $\triangle ABC$, we have*

$$\rho_a=\frac{\Delta-(p-b)(p-c)}{p}. \tag{2}$$

Similar formulas hold for ρ_b and ρ_c.

Proof. Using Theorem 2.1 and the well-known identities

$$\tan\frac{A}{2}=\frac{r}{p-a} \qquad \text{and} \qquad r=\frac{\Delta}{p},$$

we get

$$\rho_a=r\left(1-\frac{r}{p-a}\right)=r-\frac{r^2}{p-a}=\frac{\Delta}{p}-\frac{\Delta^2}{p^2(p-a)} \tag{3}$$

$$=\frac{\Delta}{p}-\frac{p(p-a)(p-b)(p-c)}{p^2(p-a)}=\frac{\Delta-(p-b)(p-c)}{p}. \tag{4}$$

This complete the proof. □

4. Barycentric coordinates of centers of γ_a, γ_b, γ_c

Theorem 4.1. *The barycentric coordinates of the center of γ_a are*

$$D=aS+2(p-b)(p-c)(b+c):bS-2b(p-b)(p-c):cS-2c(p-b)(p-c).$$

Proof. Let y be the distance between D and the sideline BC. Summing the areas of triangles DBC, DCA and DAB we obtain

$$ay+b\rho_a+c\rho_a=2\Delta. \tag{5}$$

Plugging (2) into (5) we get

$$ay+(b+c)\cdot\frac{\Delta-(p-b)(p-c)}{p}=S$$

$$ay+(b+c)\cdot\frac{S-2(p-b)(p-c)}{2p}=S$$

$$2pay+(b+c)S-2(p-b)(p-c)(b+c)=(a+b+c)S$$

$$2pay=2(p-b)(p-c)(b+c)+aS$$

$$ay=\frac{aS+2(p-b)(p-c)(b+c)}{2p} \tag{6}$$

By using (2) and (6), we obtain

$$D=\Delta DBC:\Delta DCA:DAB=ay:b\rho_a:c\rho_a$$
$$=aS+2(p-b)(p-c)(b+c):bS-2b(p-b)(p-c):cS-2c(p-b)(p-c)$$

which are the desired barycentric coordinates. □

Theorem 4.2. *The radical center of circles γ_a, γ_b and γ_c is the Gergonne point of $\triangle ABC$.*

Proof. Using Mathematica and the package `baricentricas.m`[3], it can be proved that

(a) the radical axis of γ_a and γ_b is $(p-a)x-(p-b)y=0$;
(b) the radical axis of γ_b and γ_c is $(p-b)y-(p-c)z=0$;
(c) the radical axis of γ_c and γ_a is $(p-a)x-(p-c)z=0$.

An easy verification shows that these radical axes concur at

$$G_e=(p-b)(p-c):(p-c)(p-a):(p-a)(p-b),$$

which is the Gergonne point of $\triangle ABC$. □

We can also give a purely geometric proof.

Proof. Let the incircle touch the sides of $\triangle ABC$ at Q_a, Q_b, and Q_c as shown in Figure 11. From Theorem 2.2, $Q_aE_a=Q_aF_a=r$. Thus, the tangents from Q_a to γ_b and γ_c are equal. Since $AD_c=AD_b$ and $D_cE_c=D_bF_b$ (Theorem 2.3), this means $AE_c=AF_b$. Hence the tangents from A to γ_b and γ_c are equal. The radical axis of circles γ_b and γ_c is the locus of points such that the lengths of the tangents to the two circles from that point are equal. The radical axis of two circles is a straight line. Therefore, the radical axis of circles γ_b and γ_c is AQ_a, the Gergonne cevian from A.

Similarly, the radical axis of circles γ_a and γ_c is the Gergonne cevian from B and the radical axis of circles γ_a and γ_b is the Gergonne cevian from C. Hence, the radical center of the triad of circles is the intersection point of the three Gergonne cevians, namely, the Gergonne point of $\triangle ABC$. □

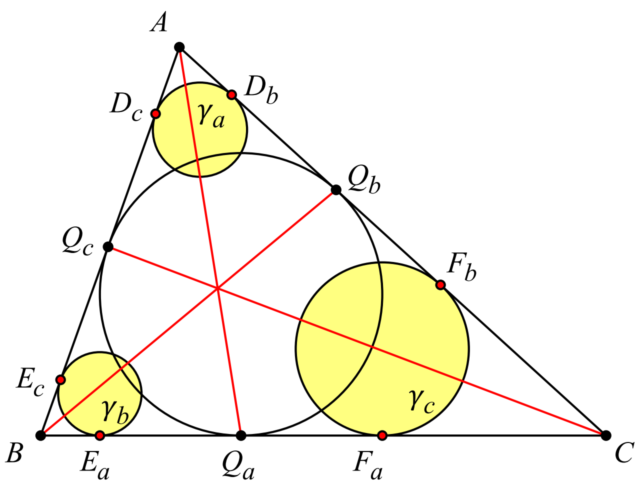

Figure 11.

[3]The package `baricentricas.m` written by F.J.G.Capitan can be freely downloaded from `http://garciacapitan.epizy.com/baricentricas/`

5. A concurrence at the Paasche point.

In [4] the following result is stated.

Theorem 5.1. *Suppose that ABC is an acute triangle. Let γ_a be the circle touching CA and AB from inside of ABC and also externally tangent to the semicircle of diameter BC, in point T_a. Define T_b and T_c cyclically (Figure 12). Then ABC is perspective to $T_aT_bT_c$, and the perspector is X_{1123}, the Paasche point of $\triangle ABC$.*

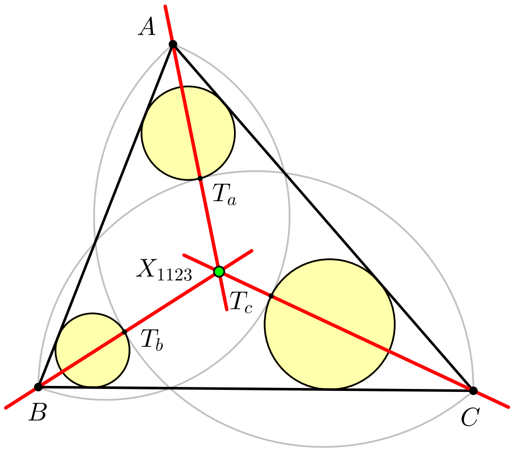

FIGURE 12.

Proof. We use homogeneous barycentric coordinates with respect to the triangle ABC. Let M_a be the midpoint of BC. The point T_a divides the segment joining the centers of the circles γ_a and ω_a in the ratio $\rho_a : \frac{a}{2}$. Using theorem 4.1 we have that the sum of coordinates of D is

$$\begin{aligned} &aS + 2(p-b)(p-c)(b+c) + bS - 2b(p-b)(p-c) + cS - 2c(p-b)(p-c) \\ =&(a+b+c)S + 2(p-b)(p-c)(b+c) - 2(b+c)(p-b)(p-c) \\ =&(a+b+c)S = 2pS. \end{aligned}$$

Therefore, by writing the coordinates of M_a in the form $M_a = 0 : pS : pS$, we get

$$T_a = \frac{a}{2} \cdot D + \rho_a \cdot M_a.$$

It follows that the first coordinate of $T_a = x_a : y_a : z_a$ is given by

$$\begin{aligned} x_a &= \frac{a}{2}\left(aS + 2(p-b)(p-c)(b+c)\right) + \rho_a \cdot 0 \\ &= \frac{a}{2}\left(aS + 2(p-b)(p-c)(b+c)\right). \end{aligned}$$

In a similar way we find that

$$\begin{aligned} y_a &= \frac{a}{2}\left(bS - 2b(p-b)(p-c)\right) + \rho_a pS \\ &= \frac{1}{2}\left(S - 2(p-b)(p-c)\right)(ab + S) \end{aligned}$$

and

$$\begin{aligned} z_a &= \frac{a}{2}\left(cS - 2c(p-b)(p-c)\right) + \rho_a p S \\ &= \frac{1}{2}\left(S - 2(p-b)(p-c)\right)(ac+S). \end{aligned}$$

Hence

$$\begin{aligned} T_a = & a^2S + 2a(p-b)(p-c)(b+c) : \\ & \left(S - 2(p-b)(p-c)\right)(ab+S) : \\ & \left(S - 2(p-b)(p-c)\right)(ac+S). \end{aligned}$$

The equation of line AT_a is $z_a y + y_a z = 0$, i.e.

$$AT_a : \quad (ac+S)y - (ab+S)z = 0.$$

The cyclic substitution $a \to b$, $b \to c$, $c \to a$ gives

$$\begin{aligned} BT_b : &\quad (bc+S)x - (ab+S)z = 0, \\ CT_c : &\quad (bc+S)x - (ac+S)y = 0. \end{aligned}$$

A direct verification shows that AT_a, BT_b, and CT_c concur at the Paasche point

$$X_{1123} = (ab+S)(ac+S) : (ab+S)(bc+S) : (ac+S)(bc+S). \qquad \square$$

6. Apollonius circles of γ_a, γ_b, γ_c

In order to find the radii of the inner and outer Apollonius circles tangent to γ_a, γ_b, and γ_c we will use the method explained in [9] and some preliminary lemmas. The more complicated calculations are performed with Mathematica.

Lemma 6.1. *If $u = EF$, $v = DF$, $w = DE$ are the distances between the centers of the circles γ_a, γ_b, and γ_c, we have*

$$\begin{aligned} u^2 &= \frac{a(b+c-a)\left(a^2+ab+ac-2b^2+4bc-2c^2\right)}{(a+b+c)^2}, \\ v^2 &= \frac{b(a-b+c)\left(-2a^2+ab+4ac+b^2+bc-2c^2\right)}{(a+b+c)^2}, \\ w^2 &= \frac{c(a+b-c)\left(-2a^2+4ab+ac-2b^2+bc+c^2\right)}{(a+b+c)^2}. \end{aligned}$$

Proof. From Theorem 3.3, we have

$$\rho_a = \frac{\Delta - (p-b)(p-c)}{p},$$

and similarly

$$\rho_b = \frac{\Delta - (p-a)(p-c)}{p}, \qquad \rho_c = \frac{\Delta - (p-a)(p-b)}{p}.$$

Assume, without loss of generality, that $\rho_b < \rho_c$. Let E' be the foot of the perpendicular from E to FF_a. Applying the Pythagorean Theorem to triangle

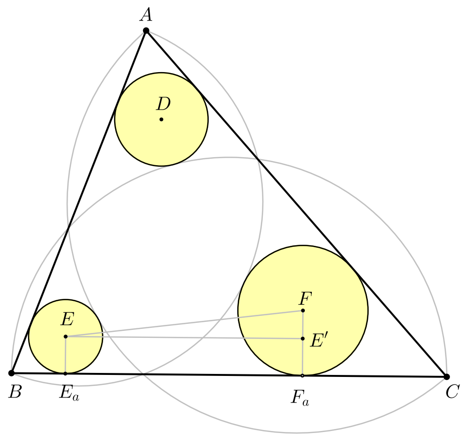

FIGURE 13.

EFE' (see figure 13), taking into account that $EE' = E_aF_a = 2r$ and $FE' = \rho_c - \rho_b$, we obtain

$$\begin{aligned}
u^2 = EF^2 &= E_aF_a^2 + (FF_a - EE_a)^2 = (2r)^2 + (\rho_c - \rho_b)^2 \\
&= 4r^2 + \left(\frac{(p-a)(p-c)-(p-a)(p-b)}{p}\right)^2 \\
&= 4\cdot\frac{\Delta^2}{p^2} + \frac{(p-a)^2(b-c)^2}{p^2} \\
&= \frac{4p(p-a)(p-b)(p-c)+(p-a)^2(b-c)^2}{p^2} \\
&= \frac{a(b+c-a)\left(a^2+ab+ac-2b^2+4bc-2c^2\right)}{(a+b+c)^2}.
\end{aligned}$$

The formulas relating to v^2 and w^2 can be proved in a similar way. □

Lemma 6.2. *If θ, φ, ψ are three positive real numbers such that $\theta+\varphi+\psi = 360°$, then we have*

$$\cos^2\theta + \cos^2\varphi + \cos^2\psi - 2\cos\theta\cos\varphi\cos\psi = 1.$$

Proof. Using the addition formulas we have

$$\begin{aligned}
&\cos^2\theta + \cos^2\varphi + \cos^2\psi - 2\cos\theta\cos\varphi\cos\psi \\
=&\cos^2\theta + \cos^2\varphi + \cos^2(360° - \theta - \varphi) - 2\cos\theta\cos\varphi\cos(360° - \theta - \varphi) \\
=&\cos^2\theta + \cos^2\varphi + \cos^2(\theta+\varphi) - 2\cos\theta\cos\varphi\cos(\theta+\varphi) \\
=&\cos^2\theta + \cos^2\varphi + (\cos\theta\cos\varphi - \sin\theta\sin\varphi)^2 - 2\cos\theta\cos\varphi\,(\cos\theta\cos\varphi - \sin\theta\sin\varphi) \\
=&\cos^2\theta + \cos^2\varphi + \cos^2\theta\cos^2\varphi + \sin^2\theta\sin^2\varphi - 2\cos^2\theta\cos^2\varphi \\
=&\cos^2\theta + \cos^2\varphi + \sin^2\theta\sin^2\varphi - \cos^2\theta\cos^2\varphi \\
=&\cos^2\theta(1-\cos^2\varphi) + \cos^2\varphi + \sin^2\theta\sin^2\varphi \\
=&\sin^2\varphi(\cos^2\theta + \sin^2\theta) + \cos^2\varphi \\
=&\sin^2\varphi + \cos^2\varphi = 1.
\end{aligned}$$

□

Theorem 6.1. *Let ρ_i be the radius of the inner Apollonius circle externally tangent to γ_a, γ_b, and γ_c (see figure 14). Then*

$$\rho_i^2 = \rho_a^2 + \rho_b^2 + \rho_c^2.$$

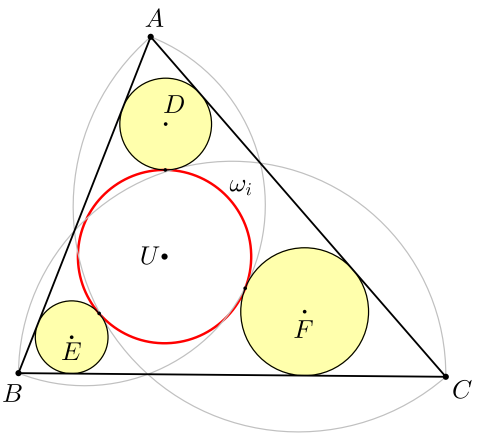

Figure 14.

Proof. Let U be the center of the inner Apollonius circle and let $x = \rho_i$. Let us consider the angles $\varphi_a = \angle EUF$, $\varphi_b = \angle FUD$, and $\varphi_c = \angle DUE$.

Since $\varphi_a + \varphi_b + \varphi_c = 360°$ by Lemma 6.2 we have

$$\cos^2\varphi_a + \cos^2\varphi_b + \cos^2\varphi_c - 2\cos\varphi_a\cos\varphi_b\cos\varphi_c = 1. \tag{7}$$

If we substitute

$$t_a = \sin^2\frac{\varphi_a}{2}, \qquad t_b = \sin^2\frac{\varphi_b}{2}, \qquad t_c = \sin^2\frac{\varphi_c}{2} \tag{8}$$

in (7), we obtain

$$t_a^2 + t_b^2 + t_c^2 - 2\left(t_at_b + t_bt_c + t_ct_a\right) + 4t_at_2t_c = 0. \tag{9}$$

Since $UD = x + \rho_a$, $UE = x + \rho_b$, $UF = x + \rho_c$, the Law of Cosines yields

$$\cos\varphi_a = \frac{(x+\rho_b)^2 + (x+\rho_c)^2 - u^2}{2(x+\rho_b)(x+\rho_c)} \quad \Rightarrow$$

$$t_a = \frac{1-\cos\varphi_1}{2} = \frac{u^2 - (\rho_b-\rho_c)^2}{4(x+\rho_b)(x+\rho_c)}, \tag{10}$$

and analogously

$$t_b = \frac{v^2 - (\rho_c-\rho_a)^2}{4(x+\rho_c)(x+\rho_a)}, \qquad t_c = \frac{w^2 - (\rho_a-\rho_b)^2}{4(x+\rho_a)(x+\rho_b)}. \tag{11}$$

Plugging (10) and (11) in (9) and using Lemma 6.1, after a straightforward calculation, we get an equation of the form $f(x)g(x) = 0$, where

$$f(x) = 6(a+b+c)x + 2ab + 2bc + 2ac - a^2 - b^2 - c^2 + 12\Delta \tag{12}$$

and

$$g(x) = 2(a+b+c)x + a^2 + b^2 + c^2 - 2ab - 2ac - 2bc + 4\Delta. \tag{13}$$

The root of (12) is

$$x = \frac{a^2+b^2+c^2-2ab-2bc-2ca-12\Delta}{6(a+b+c)} = -\frac{2r(4R+r)+6\Delta}{6p} < 0.$$

The root of (13) is

$$x = \frac{-a^2 - b^2 - c^2 + 2ba + 2bc + 2ca - 4\Delta}{2(a+b+c)} = \frac{r(4R + r - p)}{p} > 0.$$

Therefore, discarding the negative root, we have $\rho_i = \frac{r(4R+r-p)}{p}$. Hence, taking into account equation (1), we get

$$\rho_i^2 = \rho_a^2 + \rho_b^2 + \rho_c^2.$$ □

Corollary 6.1. *In Figure 15 showing the triad of circles associated with $\triangle ABC$ and the inner Apollonius circle externally tangent to each circle in the triad, we have that the sum of the yellow areas is equal to the green area.*

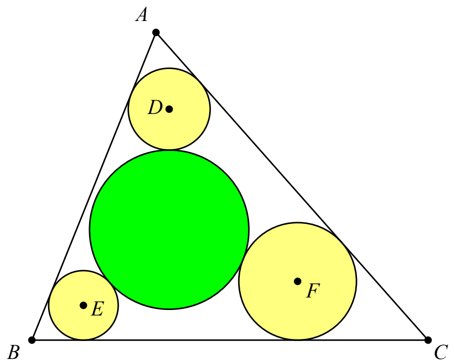

FIGURE 15. yellow area = green area

Theorem 6.2. *Let ρ_o be the radius of the outer Apollonius circle internally tangent to γ_a, γ_b, γ_c (see Figure 16). We have*

$$\rho_o = \frac{2}{3}(\rho_a + \rho_b + \rho_c) + \sqrt{\rho_a^2 + \rho_b^2 + \rho_c^2}.$$

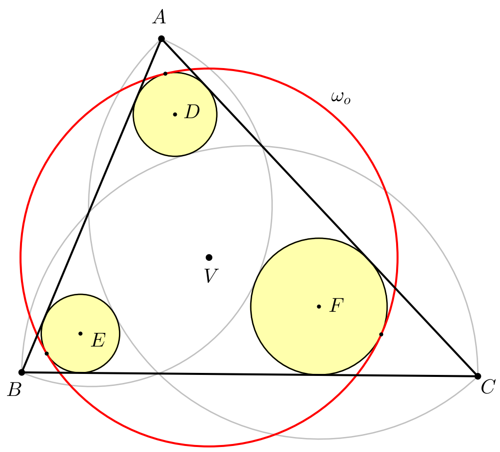

FIGURE 16.

Proof. The proof is similar to that of Theorem 6.1. □

Combining Theorem 1.1, Theorem 6.1, and Theorem 6.2, we get the following nice results.

Corollary 6.2. *The inradius* r *and the radii* ρ_i, ρ_o *of the Apollonius circles satisfy the relation*

$$3\rho_o = \rho_i + 4r.$$

Corollary 6.3. *The radii* ρ_i, ρ_o *of the Apollonius circles satisfy the relation*

$$3\rho_o = 2(\rho_a + \rho_b + \rho_c) + 3\rho_i.$$

Remark. The centers U and V of the inner and outer Apollonius circles of γ_a, γ_b, and γ_c are known ETC centers, namely $U = X(52805)$ and $V = X(52806)$.

Theorem 6.3. *Let* ω_i *be the inner Apollonius circle, externally tangent to* γ_a, γ_b, *and* γ_c. *Let* U_a *be the touch point between* ω_i *and* γ_a. *Define* U_b *and* U_c *cyclically. Let* ω_o *be the outer Apollonius circle, internally tangent to* γ_a, γ_b, *and* γ_c. *Let* V_a *be the touch point between* ω_o *and* γ_a. *Define* V_b *and* V_c *cyclically. Let* G_e *be the Gergonne point of* $\triangle ABC$. *Then the points* A, V_a, U_a, *and* G_e *are collinear. Similarly, the points* B, V_b, U_b, *and* G_e *are collinear; and the points* C, V_c, U_c, *and* G_e *are collinear (Figure 17).*

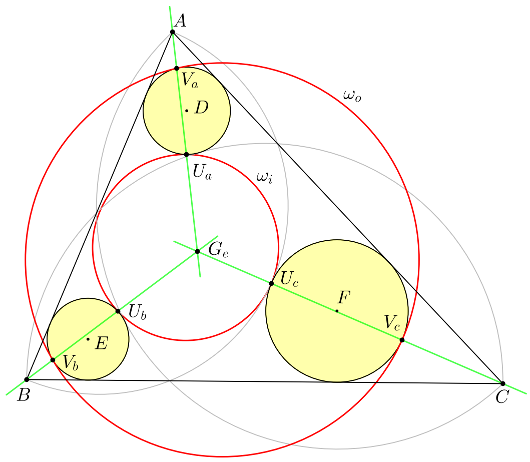

FIGURE 17.

Proof. Clearly, by symmetry, it is enough to prove that A, V_a, U_a, and G_e are collinear. From Theorem 4.2, we know that G_e is the radical center of γ_a, γ_b, and γ_c. Hence, from the Gergonne construction of Apollonius circles, it follows that U_a, V_a, and G_e are collinear. Therefore, it remains to prove that A, U_a, and G_e are collinear. To this end we use barycentric coordinates. We have $A = 1 : 0 : 0$ and $G_e = \frac{1}{p-a} : \frac{1}{p-b} : \frac{1}{p-c}$. The point U_a divides the segment DU joining the centers of the circles γ_a and ω_i in the ratio $\rho_a : \rho_i$. By using MATHEMATICA, we find that

$$U_a = \frac{-a}{(p-b)(p-c)-\Delta} : \frac{1}{p-b} : \frac{1}{p-c}.$$

The points A, U_a, and G_e are collinear because

$$\begin{vmatrix} 1 & 0 & 0 \\ \frac{-a}{(p-b)(p-c)-\Delta} & \frac{1}{p-b} & \frac{1}{p-c} \\ \frac{1}{p-a} & \frac{1}{p-b} & \frac{1}{p-c} \end{vmatrix} = 0.$$

This completes the proof. ∎

Remark. We could also show that the lines AU_a, BU_b, and CU_c are concurrent by using Theorem 2 of [8]. That theorem also shows that the point of concurrence, G_e, is the internal center of similitude of the incircle of $\triangle ABC$ and the circle ω_i.

Corollary 6.4. *Let U_a be the touch point between γ_a and ω_i (Figure 18). Then the tangents from U_a to γ_b and γ_c are equal.*

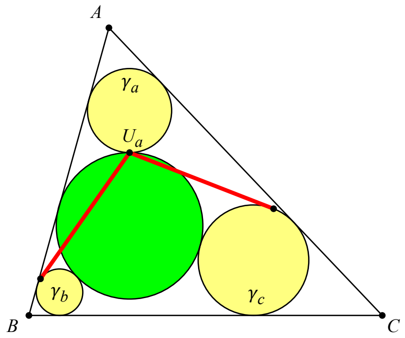

FIGURE 18. red tangents are equal

Proof. From Theorem 6.3, AU_a is the Gergonne cevian from vertex A. But from the proof of Theorem 4.2, this Gergonne cevian is the radical axis of circles γ_b and γ_c. Thus the two tangents have the same length. □

Theorem 6.4. *Let $\omega_i = (U, \rho_i)$, $\omega_o = (V, \rho_o)$ be the inner and outer Apollonius circles externally and internally tangent to γ_a, γ_b, and γ_c, respectively. Let U_a, V_a be the touch point of γ_a with ω_i and ω_o respectively. Define U_b, U_c, V_b, V_c cyclically. Let $I = X_1$, $G_e = X_7$ be the incenter and the Gergonne points of $\triangle ABC$ respectively (Figure 19). Then U and V lie on the Soddy line IG_e and $UI : IV = 3$.*

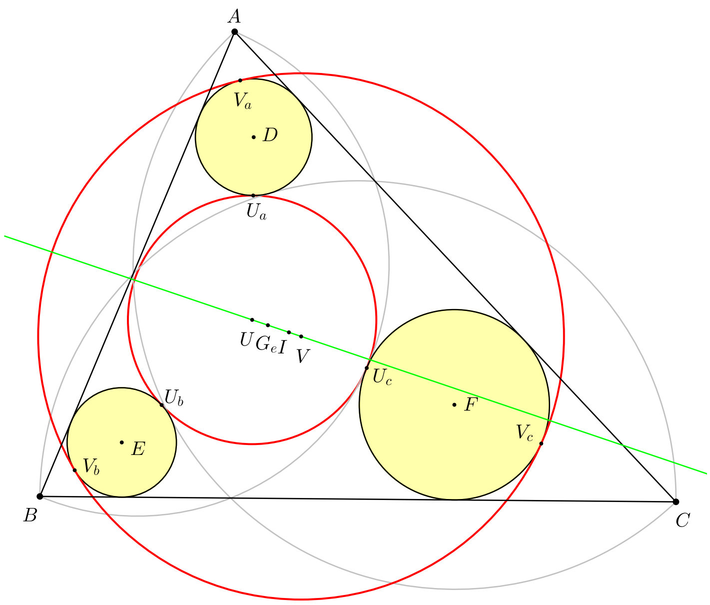

FIGURE 19.

Proof. This follows directly from the barycentric coordinates for U and V. □

7. Other properties

Theorem 7.1. *For the triad of circles associated with $\triangle ABC$, let $x = \rho_a$, $y = \rho_b$, and $z = \rho_c$. Let u, v, w be the radii of the greatest circles inscribed in the circular segments shown in Figure 20. Then*

$$xw + yu + zv = xv + zu + yw.$$

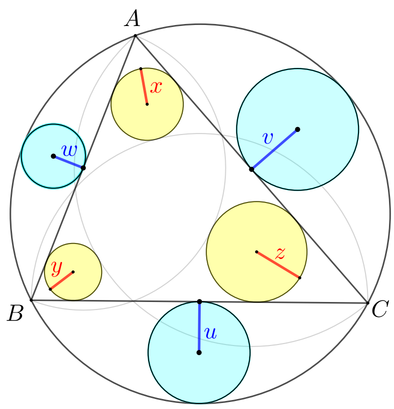

FIGURE 20.

Proof. From Theorem 2.1, we have

$$x = r\left(1 - \tan\frac{A}{2}\right), \quad y = r\left(1 - \tan\frac{B}{2}\right), \quad z = r\left(1 - \tan\frac{C}{2}\right).$$

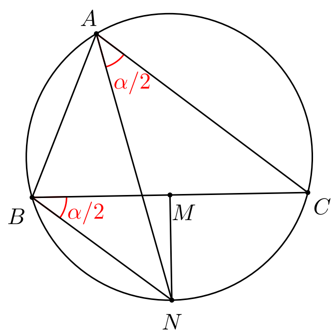

FIGURE 21.

On the other hand, we have (see Figure 21)

$$u = \frac{1}{2}MN = \frac{1}{2}BM \cdot \tan\frac{A}{2} = \frac{a}{4}\tan\frac{A}{2}$$

and similarly

$$v = \frac{b}{4}\tan\frac{B}{2}, \quad w = \frac{c}{4}\tan\frac{C}{2}.$$

Observe that

$$\tan\frac{A}{2} = \frac{r}{s-a} = \frac{\Delta}{s(s-a)}, \quad \tan\frac{B}{2} = \frac{\Delta}{s(s-b)}, \quad \tan\frac{C}{2} = \frac{\Delta}{s(s-c)}$$

so, using the Heron formula $\Delta^2 = s(s-a)(s-b)(s-c)$, we get

$$\begin{aligned} u(y-z) &= \frac{a}{4}\tan\frac{A}{2}\left(r\left(1-\tan\frac{B}{2}\right) - r\left(1-\tan\frac{C}{2}\right)\right) \\ &= \frac{ar}{4}\tan\frac{A}{2}\left(\tan\frac{C}{2}-\tan\frac{B}{2}\right) \\ &= \frac{ar}{4}\left(\tan\frac{A}{2}\tan\frac{C}{2}-\tan\frac{A}{2}\tan\frac{B}{2}\right) \\ &= \frac{ar}{4}\left(\frac{\Delta}{s(s-a)}\frac{\Delta}{s(s-c)} - \frac{\Delta}{s(s-a)}\frac{\Delta}{s(s-b)}\right) \\ &= \frac{ar}{4}\left(\frac{s-b}{s}-\frac{s-c}{s}\right) \\ &= \frac{r}{4}\cdot\frac{ac-ab}{s}. \end{aligned}$$

Similarly, we have

$$v(z-x) = \frac{r}{4}\cdot\frac{ba-bc}{s}, \qquad w(x-y) = \frac{r}{4}\cdot\frac{cb-ca}{s}.$$

Therefore

$$\begin{aligned} xw+yu+zv-(v+zu+yw) =& u(y-z)+v(z-x)+w(x-y) \\ =& \frac{r}{4}\cdot\frac{ac-ab}{s}+\frac{r}{4}\cdot\frac{ba-bc}{s}+\frac{r}{4}\cdot\frac{cb-ca}{s} \\ =& \frac{r}{4}\cdot\frac{ac-ab+ba-bc+cb-ca}{s} = 0. \end{aligned}$$

This completes the proof. □

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Errata. On page 66, in equation (9), "t_2" should be "t_b" and in equation (10), "$\cos\varphi_1$" should be "$\cos\varphi_a$".