Triangle Centers and Vertex-Distance Ordering

Stanley Rabinowitz

545 Elm St Unit 1, Milford, New Hampshire 03055, USA
e-mail: stan.rabinowitz@comcast.net

Abstract

Let P be a triangle center of $\triangle ABC$, with the usual notation $a = BC$, $b = CA$, and $c = AB$. A simple but surprisingly common phenomenon is that the three vertex distances can run in the opposite order from the side lengths. For the incenter I, for example,

$$AI^2 - BI^2 = c(b - a),$$

so $a < b$ implies $AI > BI$. We ask which triangle centers have this property. A barycentric formula for $PA^2 - PB^2$ gives an exact criterion for general center functions. It also shows that the line OP, where O is the circumcenter, is often the more natural object: one ray reverses the side ordering and the opposite ray preserves it. This viewpoint explains the Euler line and Brocard axis, classifies two natural one-parameter families, and gives a sharp criterion for a natural cone of quadratic center functions. For polynomial centers, Agaoka's decomposition reduces the problem to a one-variable root-location criterion.

Keywords. triangle center, barycentric coordinates, vertex distance, circumcenter, order-separating line, polynomial triangle center.

2020 Mathematics Subject Classification. 51M04, 51A20.

1 Introduction

Let ABC be a nondegenerate triangle. As usual,

$$a = BC, \qquad b = CA, \qquad c = AB.$$

Many familiar lengths associated with the vertices have a predictable monotonicity with respect to a, b, c; medians, altitudes, and angle bisectors provide classical examples. Monotonicity questions of this type have been studied systematically; see, for example, Mansour and Shattuck [6]. Here we ask a different question. If P is a triangle center, when does the ordering of the three distances PA, PB, PC reverse the ordering of the opposite sides?

Definition 1.1. Throughout, the following terms concern scalene triangles for which the center P is defined and finite. We call P *reverse-ordering* if

$$a < b \iff PA > PB$$

for every such triangle, and *same-ordering* if

$$a < b \iff PA < PB.$$

If $PA = PB = PC$ for every such triangle, we call P *equidistant*. A center having none of these three behaviors is said to have *no fixed ordering*.

Relabeling the vertices relabels the corresponding distances in the same way. Thus the reverse-ordering condition may equivalently be written

$$a < b < c \quad \Longrightarrow \quad PA > PB > PC,$$

and similarly for same ordering.

This phenomenon occurs for several classical centers. For the incenter I, for instance,

$$AI^2 - BI^2 = c(b-a),$$

so $a < b$ implies $AI > BI$. For the centroid G,

$$AG^2 - BG^2 = \frac{b^2 - a^2}{3}.$$

The circumcenter O is the obvious neutral example, since $OA = OB = OC$.

Our original motivation came from the electrostatic center X_{5626} of Abraham and Kovač [2]. Let $P = X_{5626}$. Their characterization gives a positive parameter q for which

$$PB + PC = a\coth(aq), \quad PC + PA = b\coth(bq), \quad PA + PB = c\coth(cq).$$

Since $x\coth(qx)$ is strictly increasing for $x > 0$, one immediately obtains $a < b \Longleftrightarrow PA > PB$. The same property is therefore present in a center whose coordinates are not especially convenient, which suggests that the phenomenon is structural rather than accidental.

There is also an extremal example. If P is any point satisfying the equal-detour relations

$$PA + PB - c = PB + PC - a = PC + PA - b,$$

then subtraction gives

$$PA - PB = b - a.$$

Thus any such point satisfies the strongest possible linear form of the reversal property. Existence and uniqueness questions for equal-detour points were studied by Veldkamp [7] and corrected by Hajja and Yff [3].

Agaoka's polynomial theory of triangle centers provides a useful algebraic setting for the polynomial case [1]. In Section 8 we show that his decomposition turns the present ordering problem into a one-variable root-location problem.

The key observation of this paper is that the difference of two squared vertex distances is affine-linear in the point P. This leads first to a simple barycentric criterion and then to a more natural theory of lines through the circumcenter.

2 The basic barycentric formula

Let

$$P = (x : y : z)$$

be a finite point in barycentric coordinates, and let

$$S = x + y + z.$$

We do not assume here that P is a triangle center.

Theorem 2.1. *For any finite barycentric point $P = (x : y : z)$,*

$$PA^2 - PB^2 = \frac{c^2(y-x) + (b^2 - a^2)z}{x+y+z}.$$

Proof. Place

$$A = (0,0), \qquad B = (c,0),$$

and

$$C = \left(\frac{b^2 + c^2 - a^2}{2c}, \frac{2\Delta}{c}\right),$$

where Δ is the area of ABC. With normalized barycentrics $X = x/S$, $Y = y/S$, $Z = z/S$, we have

$$P = XA + YB + ZC.$$

If $P = (P_x, P_y)$ in Cartesian coordinates, then

$$PA^2 - PB^2 = 2cP_x - c^2.$$

Substituting the x-coordinate of P and simplifying gives the stated formula. $\square$

The geometric content is equally simple: $PA = PB$ is the perpendicular bisector of AB, which passes through O. Thus, when $a < b$, the inequality $PA > PB$ says that P and C lie on the same side of this perpendicular bisector.

3 A criterion for barycentric center functions

We use Kimberling's center-function framework [4]. Let f be homogeneous and symmetric in its last two variables, and write

$$P_f = (f_a : f_b : f_c),$$

where

$$f_a = f(a,b,c), \quad f_b = f(b,c,a), \quad f_c = f(c,a,b).$$

The symmetry condition gives $f_b = f(b,a,c)$.

For $a \neq b$, define

$$\Lambda_f(a,b,c) = \frac{c^2(f_b - f_a) + (b^2 - a^2)f_c}{b-a}.$$

Equivalently,

$$\Lambda_f(a,b,c) = (a+b)f_c + c^2\frac{f_b - f_a}{b-a}.$$

Thus Λ_f is simply the numerator that controls the sign of $PA^2 - PB^2$, after the obvious factor $b - a$ has been removed. If f is a polynomial, then $f_b - f_a$ is divisible by $b - a$, so Λ_f is itself a polynomial in a, b, c.

Theorem 3.1. *Suppose $f_a + f_b + f_c \neq 0$. Then*

$$PA^2 - PB^2 = \frac{(b-a)\Lambda_f(a,b,c)}{f_a + f_b + f_c}.$$

Consequently, if $f_a + f_b + f_c > 0$ for every scalene triangle under consideration, then P_f is reverse-ordering if and only if

$$\Lambda_f(a,b,c) > 0$$

whenever $a < b$.

Proof. Apply Theorem 2.1 to $(f_a : f_b : f_c)$ and factor out $b - a$. $\square$

The theorem gives a useful sufficient condition almost for free.

Corollary 3.2. *Suppose $f > 0$ and*

$$a < b \quad \Longrightarrow \quad f(a, b, c) \leq f(b, a, c).$$

Then P_f is reverse-ordering.

Proof. If $a < b$, then $f_b - f_a \geq 0$, so both terms in the second formula for $\Lambda_f(a, b, c)$ are nonnegative and the first is strictly positive. $\square$

For example, the homogeneous family

$$(a^k : b^k : c^k), \qquad k \geq 0,$$

is reverse-ordering. More generally, the proof of the corollary only uses positivity and monotonicity; homogeneity is needed here only when one wants f itself to be a center function in Kimberling's sense.

The Spieker center shows that Corollary 3.2 is not necessary. Its barycentrics are

$$(b + c : c + a : a + b),$$

so $f(a, b, c) = b + c$ changes in the opposite direction. Nevertheless, Theorem 3.1 reduces the required inequality to $c < a + b$.

4 Lines through the circumcenter

The vertex-distance ordering of a point P depends on which side of O it lies. Points on opposite rays of the same line through O have opposite vertex-distance orderings. This suggests studying the line through O, rather than the particular point P.

Definition 4.1. A *line construction through O* is a rule assigning to each scalene triangle ABC a line $\ell(ABC)$ through its circumcenter. Such a construction is *order-separating* if, for every scalene triangle, one ray of $\ell(ABC)$ from O consists of points whose vertex distances reverse the side ordering, while the opposite ray consists of points whose vertex distances preserve it.

Lemma 4.2. *For a fixed scalene triangle, the three perpendicular bisectors of the sides divide the plane into six open sectors, one for each possible strict ordering of PA, PB, PC. If $a < b < c$, the sector containing H is the reverse-ordering sector*

$$PA > PB > PC,$$

and the opposite sector is the same-ordering sector

$$PA < PB < PC.$$

Proof. The equality $PA = PB$ holds precisely on the perpendicular bisector of AB, and similarly for the other two pairs. Hence the ordering of PA, PB, PC cannot change while P remains in one open sector. Moreover,

$$HA^2 - HB^2 = b^2 - a^2 > 0, \qquad HB^2 - HC^2 = c^2 - b^2 > 0,$$

so H lies in the reverse-ordering sector. Passing to the opposite sector reverses all three comparisons. $\square$

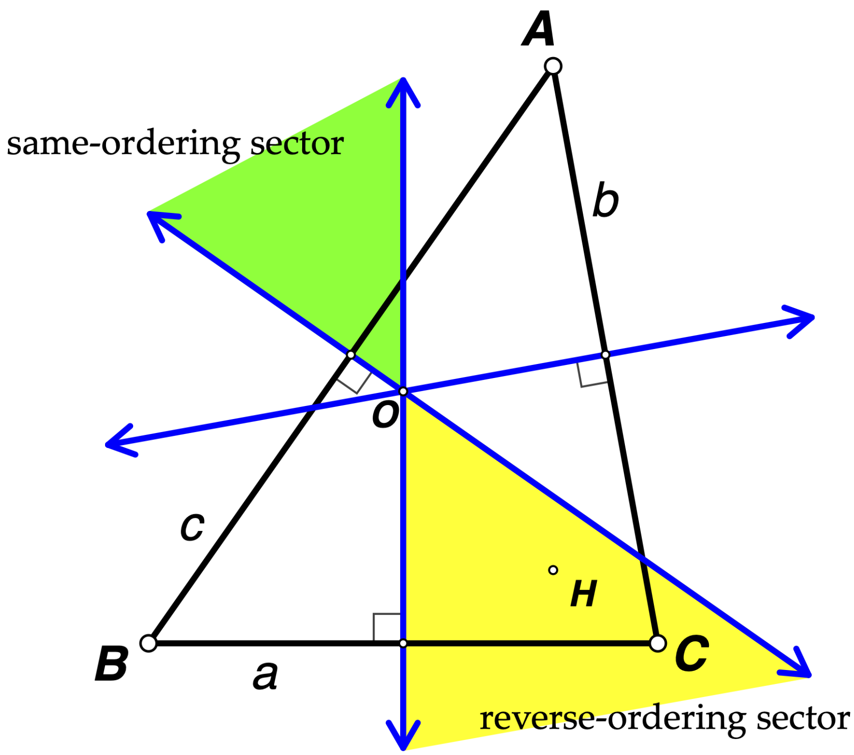

Figure 1: The three perpendicular bisectors determine the reverse-ordering sector containing H and the opposite same-ordering sector.

Figure 1 shows the two sectors singled out by Lemma 4.2. The three blue lines are the perpendicular bisectors of the sides.

Theorem 4.3. *For a fixed scalene triangle, let $Q \neq O$ and put*

$$P_t = O + t(Q - O), \qquad t \in \mathbb{R}.$$

Then

$$P_tA^2 - P_tB^2 = t(QA^2 - QB^2),$$

and cyclically. If $a < b < c$, the two rays of OQ have the reverse- and same-ordering properties if and only if

$$(QA^2 - QB^2)(QB^2 - QC^2) > 0.$$

Consequently, if $Q = Q(ABC)$ is a point construction with $Q \neq O$ for every scalene triangle, then the line construction OQ is order-separating if and only if this inequality holds for every triangle with $a < b < c$.

Proof. For fixed A, B, the function $P \mapsto PA^2 - PB^2$ is affine-linear in P. It vanishes at O, which gives the first formula on the line OQ. By Lemma 4.2, the product condition says exactly that one ray of OQ enters the reverse-ordering sector; the opposite ray then enters the same-ordering sector. □

4.1 The Euler line

Taking $Q = H$ in Theorem 4.3 gives an entire family at once.

Corollary 4.4. *If*

$$P_t = (1-t)O + tH,$$

then

$$P_tA^2 - P_tB^2 = t(b^2 - a^2).$$

Hence the ray from O through H is reverse-ordering, O is equidistant from the vertices, and the opposite ray is same-ordering.

Thus the centroid, nine-point center, orthocenter, circumcenter, and de Longchamps point are all manifestations of one line theorem rather than five separate examples. Figure 2 illustrates the reverse-ordering ray OH; the point P shown is arbitrary on this ray.

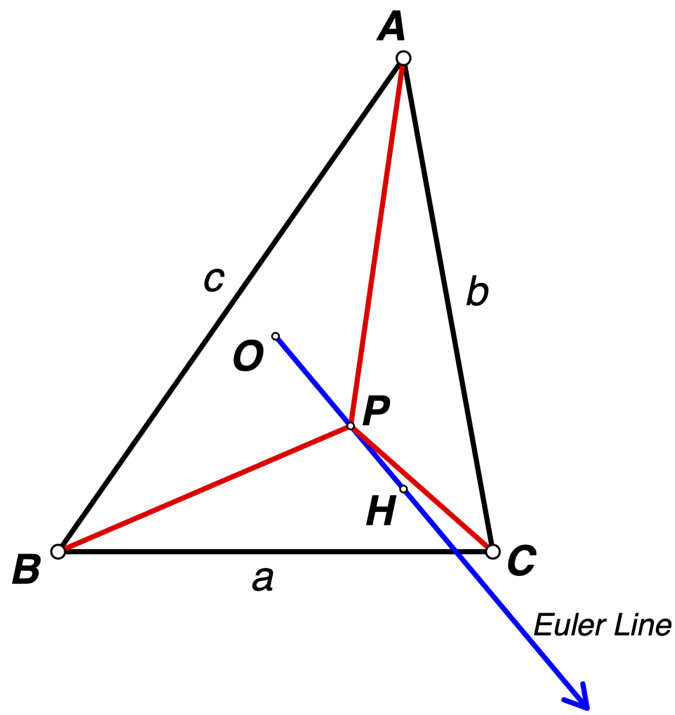

Figure 2: The ray OH is reverse-ordering: $a < b < c \Rightarrow PA > PB > PC$.

4.2 A center-function criterion for lines through O

The quantity Λ_f introduced in Section 3 is even more natural for the line problem. The numerator of $PA^2 - PB^2$ is

$$(b-a)\Lambda_f(a,b,c),$$

so the barycentric sum has disappeared.

Theorem 4.5. *Let f be a barycentric center function for which the line OP_f is defined for every scalene triangle. Then the line construction OP_f is order-separating if and only if*

$$\Lambda_f(a,b,c)\Lambda_f(b,c,a) > 0$$

for every scalene triangle with $a < b < c$. Equivalently, for every such triangle the three cyclic values of Λ_f have one common nonzero sign.

Proof. The two factors are, up to the positive factors $b-a$ and $c-b$, the numerators of $PA^2 - PB^2$ and $PB^2 - PC^2$. Theorem 4.3 now applies. The third cyclic sign follows from the identity

$$N_{AB} + N_{BC} + N_{CA} = 0,$$

where N_{AB} denotes the numerator of $PA^2 - PB^2$. $\square$

5 Some low-numbered examples

Table 1 records the behavior of the first twenty ETC centers. The entries follow from Theorem 3.1, Theorem 4.3, the two trigonometric families in the next section, or direct substitution of standard barycentrics from the Encyclopedia of Triangle Centers [5]. The table is included mainly to illustrate how common the phenomenon is; the general theory is the main point of the paper.

Center	Status	Comment
X_1	R	incenter
X_2	R	centroid
X_3	E	circumcenter
X_4	R	orthocenter
X_5	R	nine-point center
X_6	R	symmedian point
X_7	R	Gergonne point
X_8	N	Nagel point
X_9	R	Mittenpunkt
X_{10}	R	Spieker center
X_{11}	N	Feuerbach point
X_{12}	R	
X_{13}	R	first Fermat point
X_{14}	N	second Fermat point
X_{15}	R	first isodynamic point
X_{16}	R	second isodynamic point
X_{17}	R	first Napoleon point
X_{18}	N	second Napoleon point
X_{19}	R	Clawson point
X_{20}	S	de Longchamps point

Table 1: Vertex-distance ordering for $X_1, \ldots, X_{20}$. Here R means reverse ordering, S means same ordering, E means equal vertex distances, and N means no fixed ordering.

A particularly transparent failure is the Nagel point. Writing $s = (a+b+c)/2$ for the semiperimeter, its barycentrics are $(s-a : s-b : s-c)$. Theorem 2.1 gives

$$AX_8^2 - BX_8^2 = (b-a)(a+b-2c)$$

and

$$BX_8^2 - CX_8^2 = (c-b)(b+c-2a).$$

If $a < b < c$, the first expression is negative and the second is positive. Thus $AX_8 < BX_8 > CX_8$, so the three vertex distances have neither the reverse nor the same ordering.

By contrast, for the Spieker center,

$$AX_{10}^2 - BX_{10}^2 = \frac{1}{2}(b-a)(a+b-c) > 0.$$

Thus the ordinary triangle inequality is exactly what is needed.

6 Two trigonometric families

The low-numbered table contains two pairs with strikingly different behavior: X_{13} works while X_{14} does not, and X_{17} works while X_{18} does not. Both phenomena are explained by complete one-parameter classifications.

Put

$$\Sigma = \frac{a^2+b^2+c^2}{2}, \qquad S_A = \frac{b^2+c^2-a^2}{2},$$

with S_B, S_C defined cyclically. Then

$$S_A + S_B + S_C = \Sigma.$$

6.1 The family $\csc(A+\theta)$

Let P_θ have trilinear coordinates

$$\csc(A+\theta) : \csc(B+\theta) : \csc(C+\theta),$$

with the usual limiting interpretation at removable singularities. Since adding π to θ multiplies all three trilinears by -1, the parameter may be taken modulo π.

We isolate the only technical sign calculation needed below.

Lemma 6.1. *For every $u > 0$, the expression*

$$\frac{1+u^2-u(c^2/\Delta)}{1+3u^2-u(\Sigma/\Delta)}$$

takes both positive and negative values as the scalene triangle varies, away from triangles for which the denominator vanishes.

Proof. Put

$$x = \frac{c^2}{\Delta}, \qquad y = \frac{\Sigma}{\Delta}.$$

Fix C and write $r = \tan(C/2)$. Then

$$y = x + \frac{1}{r} - r,$$

and, as A varies with $A + B = \pi - C$, the scalene values of x fill $(4r, \infty)$. For x sufficiently large, numerator and denominator are both negative. On the other hand, if r is sufficiently small, then

$$3u + \frac{1}{u} + r - \frac{1}{r} < 4r < u + \frac{1}{u}.$$

Choosing x between $4r$ and $u + 1/u$ makes the numerator positive and the denominator negative. Thus both signs occur. □

Theorem 6.2. *Modulo π, the family P_θ has the following behavior:*

$$\begin{array}{c|c} 0 \le \theta \le \pi/2 & \textit{reverse ordering,} \\ \pi/2 < \theta < \pi & \textit{no fixed ordering.} \end{array}$$

No value of θ gives universal same ordering.

Proof. For $0 < \theta < \pi$, set $t = \cot\theta$; the endpoint $\theta = 0$ is obtained by continuity. Since

$$\sin(A+\theta) = \frac{\sin\theta}{bc}\big(S_A + 2\Delta t\big),$$

the barycentrics of P_θ are proportional to

$$\frac{1}{S_A + 2\Delta t} : \frac{1}{S_B + 2\Delta t} : \frac{1}{S_C + 2\Delta t}.$$

Theorem 2.1 now gives

$$P_\theta A^2 - P_\theta B^2 = (b^2 - a^2)\frac{(1+t^2)\Delta + tc^2}{(1+3t^2)\Delta + t\Sigma}. \tag{1}$$

For $t \ge 0$, numerator and denominator of the second factor are positive, so the ordering is reversed. The endpoint $\theta = 0$ is the centroid and $\theta = \pi/2$ is the orthocenter.

If $t = -u < 0$, the sign factor in (1) is exactly the expression in Lemma 6.1. Since that factor takes both signs as the triangle varies, neither reverse nor same ordering can hold universally. Hence there is no fixed ordering for $t < 0$. □

Some familiar members of this family are listed in Table 2.

$\theta \pmod{\pi}$	Center	Status
0	Centroid X_2	R
$\pi/6$	First Napoleon point X_{17}	R
$\pi/3$	First Fermat point X_{13}	R
$\pi/2$	Orthocenter X_4	R
$2\pi/3$	Second Fermat point X_{14}	N
$5\pi/6$	Second Napoleon point X_{18}	N

Table 2: Some familiar members of the $\csc(A+\theta)$ family.

6.2 The family $\sin(A+\theta)$

Let Q_θ have trilinear coordinates

$$\sin(A+\theta) : \sin(B+\theta) : \sin(C+\theta).$$

These points lie on the Brocard axis OK: after multiplication by a common factor, their barycentrics are a linear combination of barycentric representatives of the symmedian point K and the circumcenter O.

Theorem 6.3. *Modulo π, the family Q_θ has the following behavior:*

$$\begin{array}{c|c} 0 \le \theta < \pi/2 & \textit{reverse ordering,} \\ \theta = \pi/2 & \textit{equal vertex distances,} \\ \pi/2 < \theta < 2\pi/3 & \textit{no fixed ordering,} \\ 2\pi/3 \le \theta < \pi & \textit{reverse ordering,} \end{array}$$

with the usual limiting interpretation at the equilateral triangle when $\theta = 2\pi/3$. No value gives universal same ordering.

Proof. Again, for $0 < \theta < \pi$ set $t = \cot\theta$, with $\theta = 0$ understood as continuity. The same identity for $\sin(A+\theta)$ shows that the barycentrics of Q_θ are proportional to

$$a^2(S_A + 2\Delta t) : b^2(S_B + 2\Delta t) : c^2(S_C + 2\Delta t).$$

Theorem 2.1 gives

$$Q_\theta A^2 - Q_\theta B^2 = (b^2 - a^2)c^2 \frac{t}{t\Sigma + 2\Delta}. \tag{2}$$

If $t > 0$, the second factor is positive. If $t = 0$, the point is O.

Suppose $t = -u < 0$. The sign is then governed by

$$\frac{-uc^2}{2\Delta - u\Sigma}.$$

The classical Weitzenböck inequality gives

$$\frac{2\Delta}{\Sigma} \le \frac{1}{\sqrt{3}},$$

with equality only for the equilateral triangle, while the ratio tends to zero for triangles approaching degeneracy. If $0 < u < 1/\sqrt{3}$, both signs therefore occur as the triangle varies. If $u \ge 1/\sqrt{3}$, the factor in (2) is positive for every non-equilateral triangle for which the point is finite. Translating back from $u = -\cot\theta$ gives the stated intervals. □

Some familiar members of this family are listed in Table 3. In particular, the two isodynamic points occur in the same continuous Brocard-axis family.

$\theta \pmod{\pi}$	Center	Status
0	Symmedian point X_6	R
$\pi/3$	First isodynamic point X_{15}	R
$\pi/2$	Circumcenter X_3	E
$2\pi/3$	Second isodynamic point X_{16}	R

Table 3: Some familiar members of the $\sin(A+\theta)$ family.

7 Polynomial center functions

The general criterion becomes especially effective for polynomial center functions. We first record one elementary family.

Proposition 7.1. *Let*

$$f(a,b,c) = ua + v(b+c).$$

Then the line OP_f is order-separating for every scalene triangle if and only if

$$uv \ge 0.$$

Proof. A direct calculation gives

$$\Lambda_f(a,b,c) = (a+b+c)\big(uc + v(a+b-c)\big),$$

and

$$\Lambda_f(b,c,a) = (a+b+c)\big(ua + v(b+c-a)\big).$$

If $uv \geq 0$, multiply f by -1 if necessary and assume $u, v \geq 0$. Both factors are then positive by the triangle inequalities.

If $uv < 0$, scale so that $u > 0$ and $v = -ku$, $k > 0$. The two relevant factors become

$$(1+k)c - k(a+b), \qquad (1+k)a - k(b+c).$$

By taking a small and c sufficiently close to $a+b$, one can arrange for these two expressions to have opposite signs. Hence the line is not order-separating. $\square$

This one proposition contains the centroid, incenter, and Spieker center as special cases.

7.1 A quadratic cone

The next result gives a complete answer for a natural four-parameter cone of quadratic center functions.

Theorem 7.2. *Let*

$$f = \alpha a^2 + \beta a(b+c) + \gamma(b^2+c^2) + \delta bc,$$

where $\alpha, \beta, \gamma, \delta \geq 0$ and not all four coefficients vanish. Then OP_f is order-separating for every scalene triangle if and only if

$$\delta \leq 2(\alpha+\beta)$$

and

$$2\gamma + 3\delta \leq 8(\alpha+\beta).$$

Moreover, when these two inequalities hold,

$$\Lambda_f(a,b,c) > 0$$

whenever $a < b < c$. Hence, if in addition $f > 0$, the point P_f itself is reverse-ordering, by Theorem 3.1.

Proof. Put

$$a = 2h + y, \qquad b = 2h + x + y, \qquad c = 2h + x + 2y, \qquad h, x, y > 0.$$

Thus $x = b - a$, $y = c - b$, and $2h = a + b - c$. Write $M = \alpha + \beta$. Expanding $\Lambda_f(a,b,c)$ gives a homogeneous cubic in h, x, y. In the monomial order

$$x^3,\ x^2y,\ x^2h,\ xy^2,\ xyh,\ xh^2, y^3,\ y^2h,\ yh^2,\ h^3,$$

its coefficients are

$$\begin{array}{rll} 2M-\delta, & 12M-2\gamma-5\delta, & 16M-4\delta, \\ 24M-6\gamma-9\delta, & 64M-8\gamma-12\delta, & 40\alpha+44\beta+4\gamma, \\ 16M-4\gamma-6\delta, & 64M-8\gamma-12\delta, & 80\alpha+88\beta+8\gamma, \\ & 32\alpha+40\beta+16\gamma+8\delta. & \end{array}$$

If the two stated inequalities hold, every one of these coefficients is nonnegative. For example,

$$12M - 2\gamma - 5\delta = (8M - 2\gamma - 3\delta) + (4M - 2\delta),$$

and the other non-obvious cases are similar. Since the polynomial is not identically zero, it follows that

$$\Lambda_f(a, b, c) > 0 \qquad (h, x, y > 0).$$

A direct expansion of $\Lambda_f(b, c, a)$ has nonnegative coefficients for all $\alpha, \beta, \gamma, \delta \geq 0$, and again is not identically zero. Hence Theorem 4.5 applies.

Conversely, if $\delta > 2M$, let $x \to \infty$ with h, y fixed. The coefficient $2M - \delta$ of x^3 in $\Lambda_f(a, b, c)$ is negative. Since $\delta > 2M \geq 0$ forces $\delta > 0$, the coefficient $4\gamma + 2\delta$ of x^3 in $\Lambda_f(b, c, a)$ is positive. The two cyclic values therefore have opposite signs for large x.

If instead $2\gamma + 3\delta > 8M$, let $y \to \infty$. The coefficient $16M - 4\gamma - 6\delta$ of y^3 in the first cyclic value is negative. The failing inequality forces $\gamma > 0$ or $\delta > 0$, so the coefficient $6\alpha + 10\beta + 12\gamma + 5\delta$ of y^3 in the second cyclic value is positive. Again the signs are opposite for large y. In either case the line fails to be order-separating. □

Theorem 7.2 includes several familiar centers and also some less familiar ETC centers. Multiplying all three barycentric coordinates by the same nonzero symmetric factor does not change the point, so a center that is usually represented by a lower-degree function may be given a quadratic representative. The following examples use the notation of Theorem 7.2; the identifications of X_{37} and X_{86} may be found in the Encyclopedia of Triangle Centers [5].

Center	Quadratic center function $f(a, b, c)$	$(\alpha, \beta, \gamma, \delta)$
Incenter X_1	$a(a + b + c)$	$(1, 1, 0, 0)$
Centroid X_2	$(a + b + c)^2$	$(1, 2, 1, 2)$
Symmedian point X_6	a^2	$(1, 0, 0, 0)$
Spieker center X_{10}	$(b + c)(a + b + c)$	$(0, 1, 1, 2)$
X_{37}	$a(b + c)$	$(0, 1, 0, 0)$
X_{86}	$(a + b)(a + c)$	$(1, 1, 0, 1)$

Table 4: Some centers covered by Theorem 7.2.

Each displayed function is positive for every nondegenerate triangle and satisfies the two inequalities of Theorem 7.2. Thus all six centers are reverse-ordering.

Corollary 7.3. *For $\lambda \geq 0$, let $P_\lambda = P_f$, where*

$$f = a^2 + \lambda(b + c)^2.$$

Then OP_λ is order-separating if and only if

$$0 \leq \lambda \leq 1.$$

Moreover, for $0 \leq \lambda \leq 1$, the point P_λ itself is reverse-ordering.

This family has a simple geometric interpretation. The functions a^2 and $(b + c)^2$ represent X_6, the symmedian point, and X_{594}, respectively [5]. Hence P_λ lies on the line X_6X_{594}, with $P_0 = X_6$ and $P_\lambda \to X_{594}$ as $\lambda \to \infty$. Corollary 7.3 shows that precisely the range $0 \leq \lambda \leq 1$ has the universal order-separating property.

Figure 3 illustrates the X_{37} row of Table 4: the blue line is OX_{37} and the point P lies on its reverse-ordering ray.

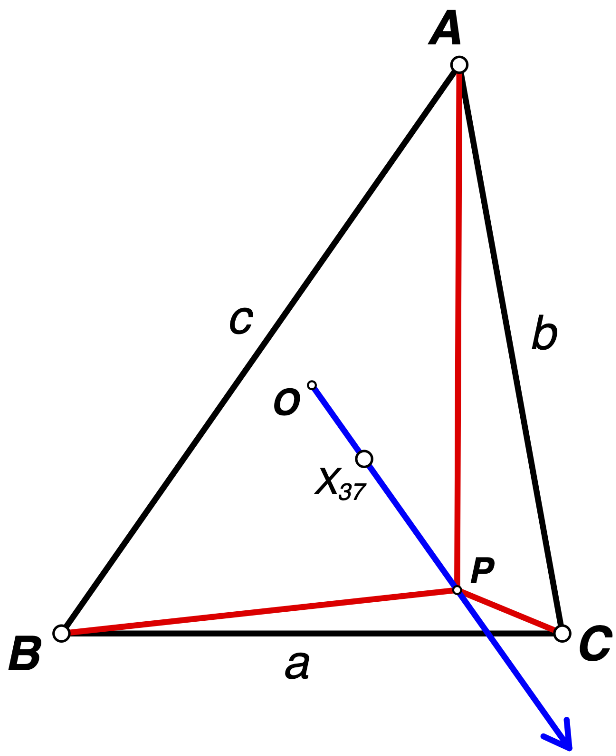

Figure 3: An example for Theorem 7.2: the ray OX_{37} is reverse-ordering.

8 Agaoka's decomposition and the Λ-operator

Agaoka [1] studies homogeneous polynomial center functions and introduces the fundamental polynomials

$$\begin{aligned}
s_1 &= a + b + c,\\
s_2 &= a^2 + b^2 + c^2 - ab - bc - ca,\\
s_3 &= (2a - b - c)(2b - c - a)(2c - a - b),\\
t_1 &= 2a - b - c,\\
t_2 &= 2a^2 - 2a(b + c) - b^2 + 4bc - c^2.
\end{aligned}$$

If T_n is the space of homogeneous degree-n polynomials satisfying $f(a, b, c) = f(a, c, b)$, and S_n is the space of homogeneous symmetric polynomials of degree n, Agaoka proves the direct-sum decomposition

$$T_n = S_n \oplus S_{n-1}t_1 \oplus S_{n-2}t_2.$$

Thus every $f \in T_n$ has a unique expression

$$f = f^{(0)} + f^{(1)}t_1 + f^{(2)}t_2, \qquad f^{(i)} \in S_{n-i}.$$

The order-separating operator has a particularly simple form in these coordinates. Put

$$t_a = 2a - b - c, \qquad t_b = 2b - c - a, \qquad t_c = 2c - a - b.$$

For $a < b < c$ we have

$$t_a < t_b < t_c.$$

The main idea is easier to see before writing the general formula. Direct calculation on the three basic pieces in Agaoka's decomposition gives

g	$3\Lambda_g(a,b,c)$
1	$2s_1 - t_c$
t_1	$s_1(s_1+4t_c)$
t_2	$-(4s_1s_2+2s_3)-(s_1^2+4s_2)t_c.$

Every entry in the right column is linear in t_c. A symmetric factor pulls through Λ, so the same is true for every polynomial center function written in Agaoka's form.

Theorem 8.1. *Let*

$$f = f^{(0)} + f^{(1)}t_1 + f^{(2)}t_2 \in T_n.$$

There is a linear function

$$L_f(t) = \mathcal{A}_f + \mathcal{B}_f t$$

such that

$$\begin{aligned} 3\Lambda_f(a,b,c) &= L_f(t_c), \\ 3\Lambda_f(b,c,a) &= L_f(t_a), \\ 3\Lambda_f(c,a,b) &= L_f(t_b). \end{aligned}$$

Explicitly,

$$\begin{aligned} \mathcal{A}_f &= 2s_1 f^{(0)} + s_1^2 f^{(1)} - (4s_1s_2+2s_3)f^{(2)}, \\ \mathcal{B}_f &= -f^{(0)} + 4s_1 f^{(1)} - (s_1^2+4s_2)f^{(2)}. \end{aligned}$$

Consequently, for a fixed triangle with $a<b<c$, the two rays of OP_f have the reverse- and same-ordering properties if and only if

$$L_f(t_a)L_f(t_c) > 0.$$

Hence OP_f is order-separating for every scalene triangle if and only if this inequality holds for every triangle with $a<b<c$.

Proof. The three displayed basis calculations, the linearity of Λ, and Agaoka's decomposition give the first identity and the formulas for $\mathcal{A}_f, \mathcal{B}_f$. The other two identities are obtained by cyclically relabeling a,b,c. Since $t_a < t_b < t_c$ and L_f is a straight line, the three values $L_f(t_a), L_f(t_b), L_f(t_c)$ have one common nonzero sign exactly when the two endpoint values have the same sign. Theorem 4.5 now completes the proof. □

Corollary 8.2. *Suppose $\mathcal{B}_f \neq 0$ and let*

$$\tau_f = -\frac{\mathcal{A}_f}{\mathcal{B}_f},$$

the zero of L_f. For a fixed triangle with $a<b<c$, the two rays of OP_f have the required opposite orderings if and only if

$$\tau_f < t_a \qquad \textit{or} \qquad \tau_f > t_c.$$

Thus OP_f is order-separating for every scalene triangle exactly when the zero τ_f lies outside $[t_a,t_c]$ for every such triangle.

Proof. A nonconstant linear function has the same sign at t_a and t_c exactly when its zero does not lie between them. Apply Theorem 8.1. □

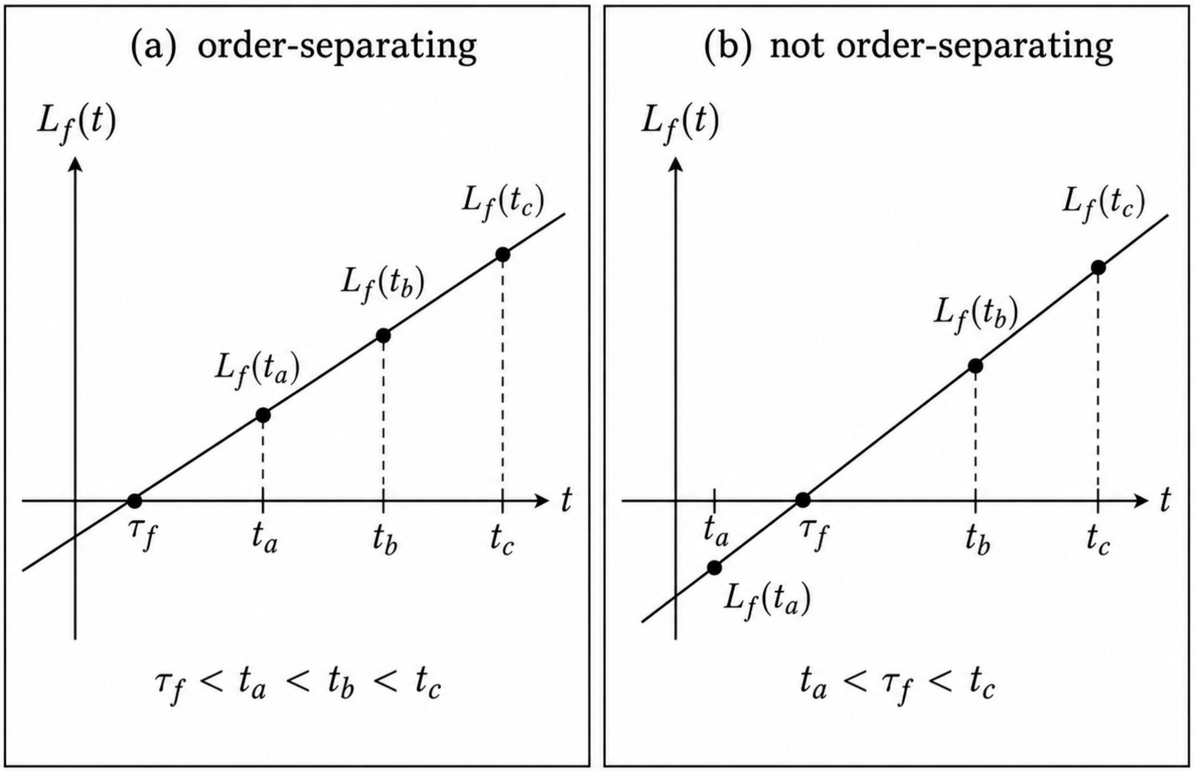

Figure 4: The root-location test. For a fixed triangle, the required two-ray ordering occurs exactly when τ_f lies outside $[t_a, t_c]$.

Figure 4 shows the root-location test. The three cyclic Λ-values are, apart from the common factor 3, simply the heights of one straight line at the ordered points t_a, t_b, t_c.

If $\mathcal{B}_f = 0$, then L_f is constant and the criterion reduces to $\mathcal{A}_f \neq 0$. If both $\mathcal{A}_f$ and $\mathcal{B}_f$ vanish, then all three Λ-values vanish. Hence, whenever P_f is finite, its three vertex distances are equal, so $P_f = O$ and no line OP_f is determined.

Example 8.3. For the centroid, $f = 1$, so

$$\mathcal{A}_f = 2s_1, \qquad \mathcal{B}_f = -1,$$

and $\tau_f = 2s_1 > t_c$.

For the incenter,

$$a = \frac{s_1 + t_1}{3},$$

so

$$\mathcal{A}_f = s_1^2, \qquad \mathcal{B}_f = s_1,$$

and $\tau_f = -s_1 < t_a$.

For the Nagel point, a polynomial representative is

$$b + c - a = \frac{s_1 - 2t_1}{3}.$$

Here $\mathcal{A}_f = 0$, hence $\tau_f = 0$. For $a < b < c$ we always have $t_a < 0 < t_c$, so the Nagel line fails for every scalene triangle. This gives a conceptual explanation for the mixed ordering observed earlier.

For the Spieker center,

$$b + c = \frac{2s_1 - t_1}{3},$$

which gives $\tau_f = s_1/2$. The inequality $\tau_f > t_c$ reduces to $a + b > c$. Thus the order-separating property of the Spieker line is exactly the triangle inequality.

Acknowledgment

The author thanks ChatGPT (OpenAI) for extensive assistance with symbolic exploration, independent verification of algebraic identities and sign conditions, literature searches, and preparation of the manuscript. The author also thanks the creators and maintainers of Clark Kimberling's Encyclopedia of Triangle Centers, which provided an indispensable source of examples and coordinate data.

The author is responsible for the final mathematical statements and presentation.

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